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2 Given that

$$u_n = \ln\left(\frac{1 + x^{n+1}}{1 + x^n}\right),$$

where $x > -1$, find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

Find the sum to infinity of the series

$$u_1 + u_2 + u_3 + \ldots$$

when

(i) $-1 < x < 1$, [1]

(ii) $x = 1$. [1]

9231_s08_ms_1

2	$u_n = \ln(1 + x^{n+1}) - \ln(1 + x)$ or for ln{Product of fractions}	B1
	$\sum_{n=1}^{N} u_n = S_N = \ln[(1 + x^{N+1})/(1 + x)]$ (AEF) Cancels → result	M1A1
	(i) $S_\infty = -\ln(1 + x)$ **OR** $\ln\left(\frac{1}{1 + x}\right)$	A1
	(ii) $S_\infty = 0$	B1

7 Prove by induction that

$$\sum_{r=1}^{n}\left(3r^5 + r^3\right) = \tfrac{1}{2}n^3(n+1)^3,$$

for all $n \geqslant 1$. [5]

Use this result together with the List of Formulae (MF10) to prove that

$$\sum_{r=1}^{n} r^5 = \tfrac{1}{12}n^2(n+1)^2\mathrm{Q}(n),$$

where $\mathrm{Q}(n)$ is a quadratic function of n which is to be determined. [3]

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7 Verifies H_1 to be true B1

$H_k : \sum_{r=1}^{k} (3r^5 + r^3) = (1/2)k^3(k+1)^3$ B1

$H_k \Rightarrow \sum_{r=1}^{k+1} (3r^5 + r^3) = (1/2)k^3(k+1)^3 + 3(k+1)^5 + (k+1)^3$ M1

$= \ldots = (1/2)(k+1)^3(k+2)^3$ A1

Thus $H_k \Rightarrow H_{k+1}$ and concludes A1

$3\sum_{r=1}^{n} r^5 + (1/4)n^2(n+1)^2 = (1/2)n^3(n+1)^3$ M1

$\Rightarrow \ldots \Rightarrow \sum_{r=1}^{n} r^5 = (1/12)n^2(n+1)^2(2n^2 + 2n - 1)$ M1A1

9 Use induction to prove that

$$\sum_{n=1}^{N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(N+1)(2N+1)}.$$ [6]

Show that

$$\sum_{n=N+1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} < \frac{3}{8N^2}.$$ [4]

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9 Set up

$$H_k : \sum_{n=1}^{k} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(k+1)(2k+1)}$$ B1

for some positive integer k

$$H_k \Rightarrow \sum_{n=1}^{k+1} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(k+1)(2k+1)} + \frac{4k+5}{(k+1)(k+2)(2k+1)(2k+3)}$$ M1

$$= 1 - \frac{2k^2+3k+1}{(k+1)(k+2)(2k+1)(2k+3)}$$ A1

$$= \ldots = 1 - \frac{1}{(k+2)(2k+3)}$$ A1

Verifies H_1 is true. B1

Correct completion of induction argument A1

$$\sum_{n=N+1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = \ldots = \frac{1}{(N+1)(2N+1)} - \frac{1}{(2N+1)(4N+1)}$$ M1A1

$$= \frac{3N}{(N+1)(2N+1)(4N+1)} < \frac{3N}{N.2N.4N} = \frac{3}{8N^2}$$ M1A1

OR

$$= \frac{3N}{8N^3+14N^2+7N+1} = \frac{3}{8N^2+14N+7+\frac{1}{N}}$$

Since $N \geqslant 1$ $14N+7+\frac{1}{N} > 0$

$$\therefore \sum < \frac{3}{8N^2}$$

2 Verify that, for all positive values of n,

$$\frac{1}{(n+2)(2n+3)} - \frac{1}{(n+3)(2n+5)} = \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}.$$ [2]

For the series

$$\sum_{n=1}^{N} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)},$$

find

(i) the sum to N terms, [3]

(ii) the sum to infinity. [1]

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2 Verifies displayed result M1A1

(i) $S_N = 1/15 - 1\,/\,(N+3)(2N+5)$ M1A1A1

(ii) $S_\infty = 1\,/\,15$ A1 ft

Note: Must see working for preliminary result

Either $(2n^2 + 11n + 15) - (2n^2 + 7n + 6)$ (oe) } in numerator

Or $\underbrace{(n+3)(2n+5)}_{\text{A1}} - \underbrace{(n+2)\,(2n+3)}_{\text{A1}}$

11 Answer only **one** of the following two alternatives.

EITHER

Prove by induction that

$$\sum_{n=1}^{N} n^3 = \tfrac{1}{4}N^2(N+1)^2. \qquad [5]$$

Use this result, together with the formula for $\sum_{n=1}^{N} n^2$, to show that

$$\sum_{n=1}^{N}(20n^3 + 36n^2) = N(N+1)(N+3)(5N+2). \qquad [3]$$

Let

$$S_N = \sum_{n=1}^{N}(20n^3 + 36n^2 + \mu n).$$

Find the value of the constant μ such that S_N is of the form $N^2(N+1)(aN+b)$, where the constants a and b are to be determined. [3]

Show that, for this value of μ,

$$5 + \frac{22}{N} < N^{-4}S_N < 5 + \frac{23}{N},$$

for all $N \geqslant 18$. [3]

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11 EITHER

$H_k : S_k = \sum_{n=1}^{k} n^3 = (1/4)k^2(k+1)^2$ for some k	B1
$H_k \Rightarrow S_{k+1} = (1/4)k^2(k+1)^2 + (k+1)^3$	M1
$= \ldots = (1/4)(k+1)^2(k+2)^2$ so that $H_k \Rightarrow H_{k+1}$	M1A1
Verifies H_1 is true and completes induction argument	A1
$\sum_{n=1}^{N}(20n^3 + 36n^2) = 5N^2(N+1)^2 + 6N(N+1)(2N+1)$	M1
$= \ldots = N(N+1)(N+3)(5N+2)$ (AG)	M1A1
$S_N = N(N+1)(N+3)(5N+2) + (\mu/2)N(N+1)$	M1
$= N(N+1)(5N^2 + 17N + 6 + \mu/2)$	M1
Take $\mu = -12$, then $S_N = N^2(N+1)(5N+17)$ so that $a = 5,\ b = 17$	A1
$N^{-4}S_N = 5 + 22/N + 17/N^2,\ > 5 + 22/N,\ \forall N \geqslant 1$	M1, A1
$N \geqslant 18 \Rightarrow N > 17 \Rightarrow 17/N^2 < 1/N$	
$\Rightarrow N^{-4}S_N < 5 + 23/N$ (AG)	A1

4 The sum S_N is defined by $S_N = \sum_{n=1}^{N} n^5$. Using the identity

$$\left(n + \tfrac{1}{2}\right)^6 - \left(n - \tfrac{1}{2}\right)^6 \equiv 6n^5 + 5n^3 + \tfrac{3}{8}n,$$

find S_N in terms of N. [You need not simplify your result.] [4]

Hence find $\lim_{N\to\infty} N^{-\lambda} S_N$, for each of the two cases

(i) $\lambda = 6$,

(ii) $\lambda > 6$.

[3]

9231_s10_ms_11

4 $(N + 1/2)^6 - 1/64 = 6S_N + (5/4)N^2(N+1)^2 + 3N(N+1)/16$ M1A1A1

M1 for application of difference method:

A1 for LHS correct: A1 for RHS correct

$S_N = (1/6)(N + 1/2)^6 - (5/24)N^2(N+1)^2 - (1/32)N(N+1) - 1/384$

Or $\frac{1}{6}\left\{(N + \frac{1}{2})^6 - (\frac{1}{2})^6 - \frac{5N^2(N+1)^2}{4} - \frac{3}{16}N(N+1)\right\}$ A1

[4]

(i) For $\lambda = 6$, $S_\infty = 1/6$ B2

(ii) For $\lambda > 6$, $S_\infty = 0$ B1

[3]

2 By considering the identity

$$\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha] \equiv 2\sin\alpha\sin 2n\alpha,$$

show that if α is not an integer multiple of π then

$$\sum_{n=1}^{N}\sin(2n\alpha) = \tfrac{1}{2}\cot\alpha - \tfrac{1}{2}\operatorname{cosec}\alpha\cos[(2N+1)\alpha]. \qquad [4]$$

Deduce that the infinite series

$$\sum_{n=1}^{\infty}\sin\left(\tfrac{2}{3}n\pi\right)$$

does not converge. [1]

9231_s10_ms_13

2 $2\sin\alpha\sum_{n=1}^{N}\sin(2n\alpha) = \cos\alpha - \cos[(2N+1)\alpha]$ M1A1

$\Rightarrow$ displayed result (AG) M1A1

[4]

$\cos(2N+1)\pi/3$ oscillates finitely as $n\to\infty \Rightarrow \sum_{n=1}^{\infty}\sin(2n\pi/3)$ does not converge (CWO) B1

Require $\alpha = \dfrac{\pi}{3}$, 'oscillate' or values of $\cos(2N+1)\dfrac{\pi}{3}$ given as $\dfrac{1}{2}$ or -1 [1]

Back **9231_s10_qp_13**

3 The sequence $x_1, x_2, x_3, \dots$ is such that $x_1 = 3$ and

$$x_{n+1} = \frac{2x_n^2 + 4x_n - 2}{2x_n + 3}$$

for $n = 1, 2, 3, \dots$. Prove by induction that $x_n > 2$ for all n. [6]

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3	$H_k : x_k > 2$ for some k	B1
	$x_{k+1} - 2 = (2x_k^2 - 8)/(2x_k + 3)$	M1A1
	$H_k \Rightarrow 2x_k^2 - 8 > 0 \Rightarrow x_{k+1} > 2 \Rightarrow H_{k+1}$	A1
	$x_1 = 3 > 2 \Rightarrow H_1$ is true	B1 CWO
	Completion of the induction argument	A1 [6]

Alternatively for lines 2 and 3:

$x_{k+1} = x_k + \frac{1}{2} - \frac{3\frac{1}{2}}{(2x_k + 3)}$	M1A1
$H_k \Rightarrow 2x_k + 3 > 7 \Rightarrow H_{k+1}$	A1
OR $x_{k+1} = x_k + \frac{x_{k-2}}{(2x_k + 3)}$	M1A1
$x_k > 2 \Rightarrow x_{k+1} > 2$	A1
OR $x_{k+1} - x_k = \frac{x_{k-2}}{(2x_k + 3)}$	M1A1
$x_k > 2 \Rightarrow x_{k+1} > x_k > 2$	A1

Minimum conclusion is 'Hence true for $n \geqslant 1$'.

2 Use the method of differences to find S_N, where

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+2)}.$$ [4]

Deduce the value of $\lim_{N\to\infty} S_N$. [1]

9231_w10_ms_1

2 nth term is $\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$	M1A1		
$S_N = \frac{1}{2}\left[\begin{array}{l}\left(\frac{1}{N} - \frac{1}{N+2}\right) + \left(\frac{1}{N-1} - \frac{1}{N+1}\right) + \left(\frac{1}{N-2} - \frac{1}{N}\right) + \ldots \\ \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{1} - \frac{1}{3}\right)\end{array}\right]$	M1	sum of terms	
$= \frac{1}{2}\left[\frac{3}{2} - \frac{1}{N+2} - \frac{1}{N+1}\right]$	A1	after cancellation	[4]
Limit = ¾	B1√		[1]

9231_w10_qp_1

4 Prove by mathematical induction that, for all non-negative integers n, $7^{2n+1} + 5^{n+3}$ is divisible by 44. [5]

9231_w10_ms_1

4	$n = 0$: $7^1 + 5^3 = 132$ which is divisible by 44	B1		
	Assume $7^{2k+1} + 5^{k+3}$ is divisible by 44	B1		
	Consider $7^{2(k+1)+1} + 5^{(k+1)+3} = 7^2 7^{2k+1} + 5.5^{k+3}$	M1	$(k + 1)$ th term	
	$= 49(7^{2k+1} + 5^{k+3}) - 44.5^{k+3}$	M1	in appropriate form	
	which is divisible by 44	A1	convincing argument	[5]
	Alternative solution for final three marks:			
	Consider $(7^{2k+3} + 5^{k+4}) - (7^{2k+1} + 5^{k+3})$	M1		
	$= 48(7^{2k+1} + 5^{k+3}) - 44.5^{k+3}$	M1	in appropriate form	
	which is divisible by 44	A1	convincing argument	

1 Express $\frac{1}{(2r+1)(2r+3)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)}.$$ [4]

Deduce the value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r+1)(2r+3)}.$$ [1]

9231_s11_ms_11

1	Any method including cover-up rule.	$\frac{1}{(2r+1)(2r+3)} = \frac{1}{2}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right)$	B1
	Expresses all terms as differences.	$S_n = \frac{1}{2}\left(\left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) \ldots\ldots \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)\right)$	M1A1
	Finds sum.	$= \frac{1}{6} - \frac{1}{2(2n+3)}$ (acf)	A1
		$S_\infty = \frac{1}{6}$ (B0M1A1√ A0A1√ if signs reversed.)	A1

4 It is given that $\mathrm{f}(n) = 3^{3n} + 6^{n-1}$.

(i) Show that $\mathrm{f}(n+1) + \mathrm{f}(n) = 28(3^{3n}) + 7(6^{n-1})$. [2]

(ii) Hence, or otherwise, prove by mathematical induction that $\mathrm{f}(n)$ is divisible by 7 for every positive integer n. [4]

9231_s11_ms_11

4 (i)	Establishes initial result.	$f(n)+f(n+1) = 3^{3n} + 6^{n-1} + 3^{3n+3} + 6^{n}$ $= 3^{3n}(1+27) + 6^{n-1}(1+6)$ $= 28(3^{3n}) + 7(6^{n-1})$ (AG)	M1 A1
(ii)	States inductive hypothesis.	H_k: $f(k) = 7\lambda$	B1
	Proves base case.	$3^3 + 6^0 = 28 = 4 \times 7 \Rightarrow H_1$ is true	B1
	Shows $P_k \Rightarrow P_{k+1}$.	$f(k+1) + f(k) = f(k+1) + 7\lambda = 28(3^{3k}) + 7(6^{k-1})$ $= 7\mu$ $\Rightarrow f(k+1) = 7(\mu - \lambda) \therefore H_k \Rightarrow H_{k+1}$	M1
	States conclusion.	(Hence by the principle of mathematical induction H_n is) **true for all positive integers** n.	A1

9231_s11_qp_13

1 Find $2^2 + 4^2 + \dots + (2n)^2$. [2]

Hence find $1^2 - 2^2 + 3^2 - 4^2 + \dots - (2n)^2$, simplifying your answer. [3]

9231_s11_ms_13

1	Finds four times sum of first n squares.	$2^2 + 4^2 + \dots + (2n)^2 = \frac{4n(n+1)(2n+1)}{6}$	M1A1
	Subtracts eight times sum of first n squares from sum of first $2n$ squares.	$1^2 - 2^2 + 3^2 - 4^2 + \dots - (2n)^2$ $= \frac{2n(2n+1)(4n+1)}{6} - \frac{8n(n+1)(2n+1)}{6}$	M1A1
	Simplifies.	$= \frac{n(2n+1)}{3}(4n+1-4n-4) = -n(2n+1)$	A1
		Or	
		$\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n - \frac{4n(n+1)(2n+1)}{6}$	(M1A1)
		$= -2n^2 - n$	(A1)

2 Let $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Prove by mathematical induction that, for every positive integer n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}.$$ [5]

9231_s11_ms_13

2	States proposition.	Let P_n be the proposition: $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \Rightarrow \mathbf{A}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}$	
	Shows base case is true.	$\mathbf{A}^1 = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 3\times(2-1) \\ 0 & 1 \end{pmatrix} \Rightarrow P_1$ is true.	B1
		Assume P_k is true for some integer k.	B1
	Proves inductive step.	$\mathbf{A}^{k+1} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k & 3(2^k - 1) \\ 0 & 1 \end{pmatrix}$	M1
		$= \begin{pmatrix} 2^{k+1} & 3.2(2^k - 1) + 3 \\ 0 & 1 \end{pmatrix}$	
		$= \begin{pmatrix} 2^{k+1} & 3(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$	A1
	States conclusion.	Since P_1 is true and $P_k \Rightarrow P_{k+1}$, hence by PMI P_n is true $\forall$ positive integers n.	A1

1 Verify that $\frac{1}{n^2} - \frac{1}{(n+1)^2} = \frac{2n+1}{n^2(n+1)^2}$. [1]

Let $S_N = \sum_{r=1}^{N} \frac{2r+1}{r^2(r+1)^2}$. Express S_N in terms of N. [2]

Let $S = \lim_{N\to\infty} S_N$. Find the least value of N such that $S - S_N < 10^{-16}$. [3]

9231_w11_ms_13

1	Verifies result.	$\frac{1}{n^2} - \frac{1}{(n+1)^2} = \frac{n^2+2n+1-n^2}{n^2(n+1)^2} = \frac{2n+1}{n^2(n+1)^2}$ (AG)	B1
	Uses difference method	$S_N = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \ldots\ldots + \left(\frac{1}{N^2} - \frac{1}{(N+1)^2}\right)$	M1
	to sum.	$= 1 - \frac{1}{(N+1)^2}$	A1
	Considers difference between sum and sum to infinity.	$S - S_N < 10^{-16} \Rightarrow \frac{1}{(N+1)^2} < 10^{-16}$	M1
		$\Rightarrow (N+1) > 10^8$	A1
	Solves inequality.	$\Rightarrow$ *least* $N = 10^8$	A1

Back

9231_s12_qp_11

2 Prove, by mathematical induction, that, for integers $n \geqslant 2$,

$$4^n > 2^n + 3^n.$$ [5]

9231_s12_ms_11

2	(States proposition.)	(P_n: $4^n > 2^n + 3^n$)	
	Proves base case.	Let $n = 2$, $16 > 4 + 9 \Rightarrow P_2$ is true.	B1
	States inductive hypothesis.	Assume P_k is true $\Rightarrow 4^k > 2^k + 3^k$	B1
	Proves inductive step.	$4^{k+1} = 4.4^k > 4(2^k + 3^k) = 4.2^k + 4.3^k$	M1
		$> 2.2^k + 3.3^k = 2^{k+1} + 3^{k+1}$	A1
		$\therefore P_k \Rightarrow P_{k+1}$	
	States conclusion.	Hence result true, by PMI, for all integers $n \geq 2$.	A1 (CWO)

Back

9231_s12_qp_11

3 Given that $f(r) = \dfrac{1}{(r+1)(r+2)}$, show that

$$f(r-1) - f(r) = \frac{2}{r(r+1)(r+2)}.$$ [2]

Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. [3]

9231_s12_ms_11

3	Proves initial result.	$f(r-1) - f(r) = \dfrac{1}{r(r+1)} - \dfrac{1}{(r+1)(r+2)}$	M1
		$= \dfrac{r+2-r}{r(r+1)(r+2)} = \dfrac{2}{r(r+1)(r+2)}$ (AG)	A1
	Sets up method of differences.	$\displaystyle\sum_{1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{1}{1\times 2} - \frac{1}{2\times 3}\right\} \cdots\cdots$	M1A1
		$+ \dfrac{1}{2}\left\{\dfrac{1}{n(n+1)} - \dfrac{1}{(n+1)(n+2)}\right\}$	
	Shows cancellation to get result.	$= \dfrac{1}{4} - \dfrac{1}{2}\left\{\dfrac{1}{(n+1)(n+2)}\right\}$ (OE)	A1
	States sum to infinity.	$\displaystyle\therefore \sum_{1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$	A1√
	'Non hence' method for last two parts	$\dfrac{1}{r(r+1)(r-2)} = \dfrac{1}{2r} - \dfrac{1}{(r+1)} + \dfrac{1}{2(r+2)}$	(M1)
	i.e. penalty of 1 mark.	$\Rightarrow \cdots \Rightarrow$	
		$\dfrac{1}{2} - \dfrac{1}{2} + \dfrac{1}{4} \ldots + \dfrac{1}{2(n+1)} - \dfrac{1}{(n+1)} + \dfrac{1}{2(n+2)}$	(A1)
		$= \dfrac{1}{4} - \dfrac{1}{2}\left\{\dfrac{1}{(n+1)(n+2)}\right\}$ (OE)	(A1)
		$\displaystyle\therefore \sum_{1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$	(A1√)

1 Find the sum of the first n terms of the series

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\dots$$

and deduce the sum to infinity. [5]

9231_s12_ms_13

1	Finds partial fractions.	$\frac{1}{r(r+2)}=\frac{1}{2}\left\{\frac{1}{r}-\frac{1}{r+2}\right\}$	M1A1
		$\sum_{r=1}^{n}\frac{1}{r(r+2)}=$	
	Use method of differences.	$\frac{1}{2}\left\{\left[\frac{1}{n}-\frac{1}{n+2}\right]+\left[\frac{1}{n-1}-\frac{1}{n+1}\right]+\dots+\left[\frac{1}{2}-\frac{1}{4}\right]+\left[1-\frac{1}{3}\right]\right\}$	M1
	Obtains results.	$=\frac{1}{2}\left\{\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right\}$ (acf) $\Rightarrow S_{\infty}=\frac{3}{4}$	A1A1√

2 For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 1$ and $u_{r+1} = \frac{3u_r - 2}{4}$ for all r. Prove by mathematical induction that $u_n = 4\left(\frac{3}{4}\right)^n - 2$, for all positive integers n. [5]

9231_s12_ms_13

2	(States proposition.)	$(P_n : u_n = 4\left(\frac{3}{4}\right)^n - 2)$	
	Proves base case.	Let $n = 1$ $\quad 4 \times \frac{3}{4} - 2 = 3 - 2 = 1 \Rightarrow P_1$ true.	B1
	States Inductive hypothesis.	Assume P_k is true for some k.	B1
	Proves inductive step.	$u_{k+1} = \dfrac{3\left\{4\left(\frac{3}{4}\right)^k - 2\right\} - 2}{4} = 4.\frac{3}{4}.\left(\frac{3}{4}\right)^k - \frac{6+2}{4}$	M1
		$= 4.\left(\frac{3}{4}\right)^{k+1} - 2 \quad \therefore P_k \Rightarrow P_{k+1}$	A1
	States conclusion.	$\therefore$ By PMI P_n is true $\forall$ positive integers.	A1

4 Let $f(r) = r(r+1)(r+2)$. Show that

$$f(r) - f(r-1) = 3r(r+1).$$ [1]

Hence show that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. [2]

Using the standard result for $\sum_{r=1}^{n} r$, deduce that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [2]

Find the sum of the series

$$1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + \ldots + 2(n-1)^2 + n^2,$$

where n is odd. [3]

9231_w12_ms_11

4	Verifies given result.	$r(r+1)(r+2)-(r-1)r(r+1) = r(r+1)(r+2-r+1)$ $= 3r(r+1)$ (AG)	B1
	Uses method of differences to sum first series.	$\sum_{r=1}^{n} r(r+1) =$ $\frac{1}{3}\{[f(n)-f(n-1)]+[f(n-1)-f(n-2)]+\ldots+[f(1)-f(0)]\}$	M1
		$=\frac{1}{3}n(n+1)(n+2)$ (AG) (Award B1 if 'not hence'.)	A1
	Subtracts $\sum_{r=1}^{n} r$ to obtain sum of second series.	$\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} r(r+1) - \sum_{r=1}^{n} r = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2}$	M1
		$=\frac{1}{6}n(n+1)(2n+4-3) = \frac{1}{6}n(n+1)(2n+1)$ (AG)	A1
	Splits series into two series.	$(1^2+2^2+\ldots+n^2)+(2^2+4^2+\ldots+(n-1)^2) =$	M1
	Applies sum of squares formula to obtain result.	$\frac{n(n+1)(2n+1)}{6} + \frac{4\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)n}{6} = \ldots = \frac{1}{2}n^2(n+1)$	M1A1

1 Show that $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$. [4]

9231_w12_ms_13

1	Use of:	$\sum_{n+1}^{2n} = \sum_{1}^{2n} - \sum_{1}^{n}$	M1
	Use of:	$\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$	M1
	Obtains result.	$\frac{2n(2n+1)(4n+1)}{6} - \frac{n(n+1)(2n+1)}{6}$	A1
		$= \frac{1}{6}n(2n+1)(8n+2-n-1) = \frac{1}{6}n(2n+1)(7n+1)$ (AG)	A1

3 Let $S_N = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{N}{(N+1)!}$. Prove by mathematical induction that, for all positive integers N,

$$S_N = 1 - \frac{1}{(N+1)!}.$$ [5]

9231_w12_ms_13

3	Proposition.	$H_N : \ S_N = 1 - \frac{1}{(N+1)!}$	
	Proves base case.	$S_1 = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2!} \Rightarrow H_1$ is true.	B1
	States inductive hypothesis.	$H_k :$ Assume $S_k = 1 - \frac{1}{(k+1)!}$ is true.	B1
	Proves inductive step.	$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)! - (k+2) + (k+1)}{(k+2)!}$	M1
		$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+2)!} \quad \therefore H_k \Rightarrow H_{k+1}$.	A1
	States conclusion.	$\therefore$ (By PMI H_n is) true for all positive integers N.	A1

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9231_s13_qp_11

2 Prove by mathematical induction that $5^{2n} - 1$ is divisible by 8 for every positive integer n. [5]

9231_s13_ms_11

2	Proves base case.	P_n: $5^{2n} - 1$ is divisible by 8. $5^2 - 1 = 24 = 3 \times 8 \Rightarrow P_1$ is true	B1 B1
	States inductive hypothesis.	Assume P_k is true: $5^{2k} - 1 = 8\lambda$ for some k. $5^{2k+2} - 1 = 25.5^{2k} - 1 = 24.5^{2k} + 5^{2k} - 1$ $= 3 \times 8.5^{2k} + 8\lambda$	M1
	Proves inductive step.	$\therefore P_k \Rightarrow P_{k+1}$	A1
	States conclusion.	(Since P_1 is true and $P_k \Rightarrow P_{k+1}$) $\therefore$ P_n is **for every positive integer** n (by PMI).	A1

5 Use the method of differences to show that $\displaystyle\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)} = \frac{1}{6} - \frac{1}{2(2N+3)}$. [5]

Deduce that $\displaystyle\sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)} < \frac{1}{8N}$. [4]

9231_s13_ms_11

5	Finds partial fractions.	$\dfrac{1}{(2r+1)(2r+3)} = \dfrac{1}{2}\left\{\dfrac{1}{2r+1} - \dfrac{1}{2r+3}\right\}$	M1A1
		$\displaystyle\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)}$	
	Expresses terms as differences.	$= \dfrac{1}{2}\left(\dfrac{1}{3} - \dfrac{1}{5}\right) + \ldots + \dfrac{1}{2}\left(\dfrac{1}{2N+1} - \dfrac{1}{2N+3}\right)$	M1A1
	Shows cancellation.	$= \dfrac{1}{6} - \dfrac{1}{2(2N+3)}$ (AG)	A1
	Uses $\displaystyle\sum_{N+1}^{2N} = \sum_{1}^{2N} - \sum_{1}^{N}$.	$\displaystyle\sum_{N+1}^{2N} = \left(\frac{1}{6} - \frac{1}{2(4N+3)}\right) - \left(\frac{1}{6} - \frac{1}{2(2N+3)}\right)$	M1
	Applies result	$= \dfrac{1}{2}\left(\dfrac{1}{2N+3} - \dfrac{1}{4N+3}\right)$	A1
	and simplifies.	$= \dfrac{N}{(2N+3)(4N+3)}$	M1
	Deduces inequality.	$< \dfrac{N}{2N.4N} = \dfrac{1}{8N}$ (AG)	A1

1 Let $f(r) = r!(r-1)$. Simplify $f(r+1) - f(r)$ and hence find $\sum_{r=n+1}^{2n} r!(r^2+1)$. [5]

9231_s13_ms_13

1	Simplifies.	$f(r+1) - f(r) = r(r+1)! - (r-1)r!$	M1
		$= r!(r^2 + r - r + 1) = r!(r^2+1)$	A1
	Uses difference method.	$\sum_1^n = f(2) - f(1) + f(3) - f(2) + \ldots f(n+1) - f(n)$	M1
		$= n(n+1)! - 0 = n(n+1)!$	A1
	Obtains result.	$\therefore \sum_{n+1}^{2n} = 2n(2n+1)! - n(n+1)!$	A1
		(Or directly using $\sum_{n+1}^{2n} = f(2n+1) - f(n+1)$ from the method of differences.)	

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9231_w13_qp_11

3 It is given that

$$S_n = \sum_{r=1}^{n} u_r = 2n^2 + n.$$

Write down the values of S_1, S_2, S_3, S_4. Express u_r in terms of r, justifying your answer. [4]

Find

$$\sum_{r=n+1}^{2n} u_r.$$ [3]

9231_w13_ms_11

3	Writes first four sums.	$S_1 \ldots S_4 \sim 3, 10, 21, 36$	B1
	Deduces first four terms, conjectures and justifies result.	$u_1 \ldots u_4 \sim 3, 7, 11, 15 \Rightarrow u_r = 4r - 1$	B1B1
		since $S_n = \frac{n}{2}\{6 + 4(n-1)\} = 2n^2 + n$ as given.	B1
		Or $u_r = S_r - S_{r-1} = 2r^2 + r - 2(r-1)^2 - (r-1)$	B1B1
		$= 4r - 1$	B1
	Obtains required sum.	$\sum_{n+1}^{2n}(4r-1) = 4.\frac{2n(2n+1)}{2} - 2n - \left(4.\frac{n(n+1)}{2} - n\right)$	M1A1
		$= 8n^2 + 2n - (2n^2 + n) = 6n^2 + n$	A1
		Or Sum of AP $= \frac{n}{2}(4n + 3 + 8n - 1) = 6n^2 + n$	

Back

9231_w13_qp_13

1 Express $\dfrac{1}{r(r+1)(r-1)}$ in partial fractions. [1]

Find

$$\sum_{r=2}^{n} \frac{1}{r(r+1)(r-1)}.$$ [4]

State the value of

$$\sum_{r=2}^{\infty} \frac{1}{r(r+1)(r-1)}.$$ [1]

9231_w13_ms_13

1	Finds partial fractions.	$\dfrac{1}{r(r-1)(r+1)} = \dfrac{1}{2(r-1)} - \dfrac{1}{r} + \dfrac{1}{2(r+1)}$	B1
	Expresses each term in fractions	$\left(\dfrac{1}{2} - \dfrac{1}{2} + \dfrac{1}{6}\right) + \left(\dfrac{1}{4} - \dfrac{1}{3} + \dfrac{1}{8}\right) \ldots \left(\dfrac{1}{2(n-1)} - \dfrac{1}{n} + \dfrac{1}{2(n+1)}\right)$	M1A1
	Cancels terms and sums	$= \dfrac{1}{4} - \dfrac{1}{2n} + \dfrac{1}{2(n+1)}$ (OE)	M1A1
	Find sums to infinity	$S\infty = \dfrac{1}{4}$	B1

2 Expand and simplify $(r+1)^4 - r^4$. [1]

Use the method of differences together with the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2. \quad [4]$$

9231_s14_ms_11

2	$(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$	B1 [1]
	$(n+1)^4 - 1^4 = 4\Sigma_{r=1}^{n} r^3 + 6\Sigma_{r=1}^{n} r^2 + 4\Sigma_{r=1}^{n} r + n$	M1
	$n^4 + 4n^3 + 6n^2 + 4n = 4\Sigma_{r=1}^{n} r^3 + n(2n^2 + 3n + 1) + 2n^2 + 2n + n$	A1A1
	$\Rightarrow \ldots \Rightarrow \Sigma_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$. (AG)	A1 [4]

3 Prove by mathematical induction that, for all non-negative integers n,

$$11^{2n} + 25^{n} + 22$$

is divisible by 24. [6]

9231_s14_ms_11

3	H_k: $f(k) = 11^{2k} + 25^{k} + 22 = 24\lambda$	B1
	$f(0) = 1 + 1 + 22 = 24 = 1 \times 24 \Rightarrow H_0$ is true.	B1
	$f(k+1) - f(k) = 11^{2k+2} + 25^{k+1} + 22 - (11^{2k} + 25^{k} + 22)$ $= 11^{2k}(121 - 1) + 25^{k}(25 - 1)$ $= 11^{2k} \times 24 \times 5 + 25^{k} \times 24 = 24\mu$	M1 A1 A1
	Alternatively:	
	$f(k+1) = 11^{2k+2} + 25^{k+1} + 22$ $= 121.11^{2k} + 25.25^{k} + 22 = (120 + 1)11^{2k} + (24 + 1)25^{k} + 22$ (OE) $= 120.11^{2k} + 24.25^{k} + 24\lambda = 24\mu$	(M1) (A1) (A1)
	$\Rightarrow f(k+1) = 24\mu + 24\lambda = 24(\mu + \lambda) \Rightarrow H_{k+1}$ is true. Hence by PMI H_n is true for all non-negative integers. (Must see non-negative integers.) CSO: Final mark requires all previous marks.	A1 [6]

2 Show that the difference between the squares of consecutive integers is an odd integer. [1]

Find the sum to n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots + \frac{2r+1}{r^2(r+1)^2} + \ldots$$

and deduce the sum to infinity of the series. [5]

9231_s14_ms_13

2	$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 \Rightarrow$ odd.	B1 [1]
	$\frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + \ldots \frac{2n+1}{n^2(n+1)^2} = \frac{2^2-1^2}{1^2.2^2} + \frac{3^2-2^2}{2^2.3^2} + \frac{4^2-3^2}{3^2.4^2} + \ldots \frac{(n+1)^2-n^2}{n^2(n+1)^2}$	M1A1
	$= 1 - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots + \frac{1}{n^2} - \frac{1}{(n+1)^2}$	M1
	$= 1 - \frac{1}{(n+1)^2}$	A1
	Sum to infinity =1.	A1√ [5]

3 It is given that $\phi(n) = 5^n(4n+1) - 1$, for $n = 1, 2, 3, \ldots$. Prove, by mathematical induction, that $\phi(n)$ is divisible by 8, for every positive integer n. [7]

9231_s14_ms_13

3	$\phi(1) = 5 \times 5 - 1 = 24$ which is divisible by 8 $\Rightarrow H_1$ is true.	B1
	Assume P_k is true for some positive integer $k \Rightarrow \phi(k) = 8l$	B1
	$\phi(k+1) - \phi(k) = 5^{k+1}(4k+5) - 1 - 5^k(4k+1) + 1$	M1
	$= 5^k(20k + 25 - 4k - 1)$	A1
	$= 5^k(16k + 24) = 8m$	A1
	$\therefore \phi(k+1) = 8(l+m)$	A1
	Hence, by PMI, true for all positive integers n. (CWO – all previous marks required.)	A1
		[7]
	Alternatively	
	$\phi(k+1) = 5^{k+1}(4k+5) - 1$	
	$= 5.(4k.5^k) + 25.5^k - 1$	
	$= 5(8l - 5^k + 1) + 25.5^k - 1$	(M1A1)
	$= 40l + 20.5^k + 4$	
	$= 40l + 24.5^k - 4.5^k + 4$	(A1)
	$= 40l + 24.5^k - 4(5^k - 1)$	
	$= 40l + 24.5^k - 4(8l - 4k.5^k)$	
	$= 8l + 24.5^k + 16k.5^k$	(A1)
	$= 8m$	

1 Given that

$$u_k = \frac{1}{\sqrt{(2k-1)}} - \frac{1}{\sqrt{(2k+1)}},$$

express $\sum_{k=13}^{n} u_k$ in terms of n. [4]

Deduce the value of $\sum_{k=13}^{\infty} u_k$. [1]

9231_w14_ms_11

1	$\left(\frac{1}{\sqrt{25}} - \frac{1}{\sqrt{27}}\right) + \left(\frac{1}{\sqrt{27}} - \frac{1}{\sqrt{29}}\right) + \ldots + \left(\frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{2n+1}}\right)$	M1A1
	$\sum_{r=13}^{n} u_k = \frac{1}{5} - \frac{1}{\sqrt{2n+1}}$	M1A1 (4)
	$\sum_{r=13}^{\infty} u_k = \frac{1}{5}$	B1√ (1) **[5]**

3 It is given that $u_r = r \times r!$ for $r = 1, 2, 3, \dots$. Let $S_n = u_1 + u_2 + u_3 + \dots + u_n$. Write down the values of

$$2! - S_1, \quad 3! - S_2, \quad 4! - S_3, \quad 5! - S_4. \qquad [2]$$

Conjecture a formula for S_n. [1]

Prove, by mathematical induction, a formula for S_n, for all positive integers n. [4]

9231_w14_ms_11

3	$2! - S_1 = 1$, $3! - S_2 = 1$, $4! - S_3 = 1$, $5! - S_4 = 1$ (Two correct B1, all four correct B2)	B2,1,0 (2)
	$S_n = (n+1)! - 1$	B1 (1)
	$2! - 1 = 2 - 1 = 1 \Rightarrow H_1$ is true.	B1
	H_k: $S_k = (k+1)! - 1$	B1
	$(k+1)! - 1 + (k+1) \times (k+1)!$ $= (k+1)!(1 + k + 1) - 1$	M1
	$= ([k+1]+1)! - 1$ Hence $H_k \Rightarrow H_{k+1}$ So result holds for all positive integers (by PMI).	A1 (4) **[7]**

1 Use the List of Formulae (MF10) to show that $\sum_{r=1}^{13}(3r^2 - 5r + 1)$ and $\sum_{r=0}^{9}(r^3 - 1)$ have the same numerical value. [4]

9231_s15_ms_11

1	$3 \times \frac{13\times14\times27}{6} - 5\times\frac{13\times14}{2} + 13 = 2015$	M1A1
	$\left[\frac{9\times10}{2}\right]^2 - 10 = 2015$ (Award M1 for subtracting 9 or 10 here.)	M1A1 (4) **Total: 4**

3 The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 5$ and $a_{n+1} = \frac{4a_n}{5} + \frac{5}{a_n}$ for every positive integer n.

Prove by mathematical induction that $a_n > 5$ for every positive integer n. [5]

Prove also that $a_n > a_{n+1}$ for every positive integer n. [2]

9231_s15_ms_11

3	$a_1 > 5$ (given) $\Rightarrow \text{H}_1$ is true.	B1
	Assume H_k is true for some positive integer k, i.e. $a_k = 5 + \delta$, where $\delta > 0$.	B1
	$a_{k+1} - 5 = \frac{4a_k^{\;2} + 25}{5a_k} - 5 = \frac{4a_k^{\;2} + 25 - 25a_k}{5a_k} = \frac{(4a_k - 5)(a_k - 5)}{5a_k} > 0 \;, \Rightarrow a_{k+1} > 5$	M1A1
	Or	
	$a_{k+1} = \frac{4}{5}(5+\delta) + \frac{5}{5+\delta}, \; = 4 + \frac{4}{5}\delta + (1 - \frac{\delta}{5} + \frac{\delta^2}{25} - \ldots)$ for $0 < \delta < 5$	(M1) (A1)
	$= 5 + \frac{3}{5}\delta + 0(\delta^2) \geqslant a_{k+1} > 5$, ($\delta \geqslant 5$ is trivial).	
	$\text{H}_k \Rightarrow \text{H}_{k+1}$ and H_1 is true, hence by mathematical induction, the result is true for all $n \in \mathbf{Z}^+$ (N.B. The minimum requirement is 'true for all positive integers'.)	A1 (5)
	$a_{k+1} - a_k = \frac{5}{a_k} - \frac{1}{5}a_k$	M1
	$\frac{5}{a_k} < 1$ and $\frac{1}{5}a_k > 1 \Rightarrow a_{k+1} - a_k < 0 \Rightarrow a_{k+1} < a_k$	A1 (2)
		Total: 7

3 Prove by mathematical induction that, for all positive integers n, $\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1} = \frac{n}{2n+1}$. [6]

State the value of $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

9231_s15_ms_13

3	$H_k : \sum_{r=1}^{k} \frac{1}{(2r)^2 - 1} = \frac{k}{2k+1}$ is true for some integer k.	B1
	$\frac{1}{2^2 - 1} = \frac{1}{3} = \frac{1}{2 \times 1 + 1} \Rightarrow H_1$ is true.	B1
	$\frac{k}{2k+1} + \frac{1}{(2k+2)^2 - 1} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$	M1A1
	$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2[k+1]+1}$	A1
	$\therefore H_k \Rightarrow H_{k+1}$ $\therefore$ (By Principle of Mathematical Induction) H_n is true for all positive integers n. (This mark requires all previous marks.)	A1 (6)
	$\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1} = \frac{1}{2}$	B1 (1) **Total 7**

9231_w15_qp_11

4 The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n,

$$a_n = \frac{n+5}{\sqrt{(n^2-n+1)}} - \frac{n+6}{\sqrt{(n^2+n+1)}}.$$

The sum $\sum_{n=1}^{N} a_n$ is denoted by S_N. Find

(i) the value of S_{30} correct to 3 decimal places, [3]

(ii) the least value of N for which $S_N > 4.9$. [4]

9231_w15_ms_11

4	**(i)**	$\left(\frac{6}{\sqrt{1}}-\frac{7}{\sqrt{3}}\right)+\left(\frac{7}{\sqrt{3}}-\frac{8}{\sqrt{7}}\right)+\ldots+\left(\frac{35}{\sqrt{871}}-\frac{36}{\sqrt{931}}\right)=6-\frac{36}{\sqrt{931}}=4.820$	**M1A1** **A1** [3]
	(ii)	$6-\frac{n+6}{\sqrt{n^2+n+1}}>4.9 \Rightarrow 0.21n^2-10.79n-34.79(>0)$	**M1A1**
		$\Rightarrow n>54.42\ldots$ so 55 terms required.	**dM1A1** [4] **Total** **7**

2 Express $\frac{4}{r(r+1)(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)}$. [5]

Deduce the value of $\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)}$. [1]

9231_s16_ms_11

2	$\frac{2}{r} - \frac{4}{r+1} + \frac{2}{r+2}$ (Award **B2** if written down by cover up rule.)	**M1A1**
	$\left(2-2+\frac{2}{3}\right)+\left(1-\frac{4}{3}+\frac{1}{2}\right)+\ldots+\left(\frac{2}{n-1}-\frac{4}{n}+\frac{2}{n+1}\right)+\left(\frac{2}{n}-\frac{4}{n+1}+\frac{2}{n+2}\right)$	**M1A1**
	$= 1 - \frac{2}{n+1} + \frac{2}{n+2}$ (AEF)	**A1** [5]
	Sum to infinity = 1	**B1**√ [1]

3 Prove by mathematical induction that, for all positive integers n, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. [6]

9231_s16_ms_11

3	For $n=1$ $10 + 192 + 5 = 207 = 9 \times 23 \Rightarrow H_1$ is true.	**B1**
	Assume H_k is true for some positive integer $k \Rightarrow 10^n + 3.4^{n+2} + 5 = 9\alpha$	**B1**
	Let $f(n) = 10^n + 3.4^{n+2} + 5$	
	Hence $f(n+1) - f(n) = 10^n(10-1) + 3.4^{n+2}(4-1)$	**M1**
	$= 9\left(10^n + 4^{n+2}\right)$	
	$= 9\beta$	**A1**
	Hence $f(n+1)\left(=9(\beta+\alpha)\right) \Rightarrow H_{k+1}$ is true	**A1**
	H_1 is true and $H_k \Rightarrow H_{k+1}$, hence by PMI H_n is true for all positive integers n.	**A1**
		[6]
	N.B. Or can show $f(n+1) = 9\left(10\alpha - 2.4^{n+2} - 5\right)$ for **M1A1A1**. (3rd ,4th&5th marks)	

1 Verify that $\dfrac{1}{(3r+1)(3r+4)} = \dfrac{1}{3}\left(\dfrac{1}{3r+1} - \dfrac{1}{3r+4}\right)$. [1]

Let S_N denote $\displaystyle\sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}$ and let S denote $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(3r+1)(3r+4)}$. Find the least value of N such that $S - S_N < \frac{1}{10\,000}$. [5]

9231_s16_ms_13

1	$\dfrac{1}{3}\left(\dfrac{1}{3r+1}-\dfrac{1}{3r+4}\right)=\dfrac{1}{3}\left(\dfrac{(3r+4)-(3r+1)}{(3r+1)(3r+4)}\right)=\dfrac{1}{(3r+1)(3r+4)}$ **AG**	**B1** [1]
	$S_N=\dfrac{1}{3}\left[\left(\dfrac{1}{4}-\dfrac{1}{7}\right)+\left(\dfrac{1}{7}-\dfrac{1}{10}\right)+\ldots+\left(\left(\dfrac{1}{3N+1}\right)-\dfrac{1}{3N+4}\right)\right]=\dfrac{1}{3}\left(\dfrac{1}{4}-\dfrac{1}{3N+4}\right)$	**M1 A1**
	$\Rightarrow S=\dfrac{1}{12}$	**A1**
	$\Rightarrow S-S_N=\dfrac{1}{3(3N+4)}<\dfrac{1}{10000}$	**M1**
	$\Rightarrow 3N+4>\dfrac{10000}{3}\Rightarrow N>1109\ \dfrac{7}{9}$. Thus least N is 1110.	**A1** [5]

2 It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that an n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals, where $n \geqslant 3$. [6]

9231_s16_ms_13

2	With $n=3$, $\frac{1}{2}n(n-3)=0$	**M1**
	A triangle has no diagonals $\Rightarrow$ H_3 is true.	**A1**
	Assume H_k is true: A k-gon has $\frac{1}{2}k(k-3)$ diagonals for some integer ≥ 3	**B1**
	Adding an extra vertex, a further $(k-1)$ diagonals can be drawn.	**M1**
	$\frac{1}{2}k(k-3)+k-1=\frac{k^2-3k+2k-2}{2}=\frac{(k+1)(k-2)}{2}$	**A1**
	$=\frac{1}{2}(k+1)(k+1-3)$ (So $H_k \Rightarrow H_{k+1}$)	**A1**
	$\Rightarrow H_n$ is true for all integers $n \geqslant 3$.	[6]

1 Use the method of differences to find $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1}$. [4]

Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

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1	$\dfrac{1}{(2r-1)(2r+1)} = \dfrac{1}{2}\left(\dfrac{1}{2r-1} - \dfrac{1}{2r+1}\right)$	**M1A1**
	$\displaystyle\sum_{r=1}^{n} \frac{1}{(2r)^2-1} = \frac{1}{2}\left(\left[1-\frac{1}{3}\right]+\left[\frac{1}{3}-\frac{1}{5}\right]+\ldots+\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]\right) = \frac{1}{2}\left(1-\frac{1}{2n+1}\right)$ (OE)	**M1A1**
	$\displaystyle\frac{1}{2}\left(1-\frac{1}{2n+1}\right) = \frac{n}{2n+1} \Rightarrow \sum_{r=1}^{\infty} \frac{1}{(2r)^2-1} = \frac{1}{2}$	**B1**√

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4 Using factorials, show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. [2]

Hence prove by mathematical induction that

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}x^n$$

for every positive integer n. [4]

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4	$\binom{n}{r-1}+\binom{n}{r} = \frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r)!}\left(\frac{1}{n-r+1}+\frac{1}{r}\right)$	**M1**
	$= \frac{n!}{(r-1)!(n-r)!}\left(\frac{r+n-r+1}{r(n-r+1)}\right) = \frac{(n+1)!}{r!(n-r+1)!} = \binom{n+1}{r}$	**A1**
	$(a+x)^1 = \binom{1}{0}a+\binom{1}{1}x = a+x \Rightarrow H_1$ is true.	**B1**
	Assume H_k is true,i.e. $(a+x)^k = \binom{k}{0}a^k+\binom{k}{1}a^{k-1}x+\ldots+\binom{k}{r}a^{k-r}x^r+\ldots+\binom{k}{k}x^k$	**B1**
	Multiplying by $(a+x)$, the coefficient of $a^{k-r+1}x^r$ is: $\binom{k}{r-1}+\binom{k}{r}=\binom{k+1}{r}$	**M1**
	$\Rightarrow H_{k+1}$ is true.	
	Hence H_n is true for all positive integers.	**A1**